Lecture 10: LTI IIR design: Analog to Digital and spectral transformations

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Techniques for analog filter design are well developed, so it is quite natural to start designing a digital filter from an analog prototype.

An analog LTI system is stable if all poles of its transfer function $H(s)$ lie in the left half of the $s$-plane. Therefore:

1) The $j\Omega$ axis of the $s$-plane should map into the unit circle in the $z$-plane.
2) The left half of the $s$-plane should map into the inside of the unit circle in the $z$-plane to convert a stable analog filter to a stable digital filter.

A linear phase filter must have a transfer function satisfying:

$$H(z) = \pm z^{-N}H(z^{-1})$$  \hspace{1cm} (10.2.1)

However, in this case, filter would have a reciprocal pole outside the UC and, therefore, would be unstable. A causal and stable IIR filter cannot have linear phase.

As a result, when designing IIR filters, only the magnitude response is specified.
Approximation of derivatives

If the analog filter is specified by the LCC Differential Equation:

\[ \sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k} \]  

(10.3.1)

the following approximation for the derivative can be used:

\[ \left. \frac{dy(t)}{dt} \right|_{t=nT} = \frac{y(nT) - y(nT-T)}{T} \]

(10.3.2)

Where \( T \) is the sampling period.

The system function of an analog differentiator:

\[ H(s) = s \]  

(10.3.3)

The system function of a digital differentiator:

\[ H(z) = \frac{1 - z^{-1}}{T} \]  

(10.4.1)

Therefore, the frequency-domain equivalent for (10.3.2) is:

\[ s = \frac{1 - z^{-1}}{T} \]  

(10.4.1)

Similarly, the \( k \)th derivative can be expressed as:

\[ s^k = \left( \frac{1 - z^{-1}}{T} \right)^k \]  

(10.4.2)

The system transfer function for the digital IIR filter can be approximated as:

\[ H(z) = H_a(s) \bigg|_{s=\frac{1 - z^{-1}}{T}} \]  

(10.4.3)

Equivalently:

\[ z = \frac{1}{1 - sT} \]  

(10.4.4)
Approximation of derivatives

However, it can be shown that this method is suitable for a quite limited class of filters due to its mapping property:

There is an attempt to overcome this limitation by using an alternative mapping:

\[
\frac{dy(t)}{dt} \bigg|_{t=nT} = \frac{1}{T} \sum_{k=0}^{1} \alpha_k \frac{y_{nT} + T - y_{nT-k}^T}{T}
\]  

(10.5.1)

Therefore:

\[
s = \frac{1}{T} \sum_{k=1}^{L} \gamma_k \left( z^k - z^{-k} \right)
\]  

(10.5.2)

By the proper selection of coefficients \( \gamma_k \), it is possible to map the \( j\Omega \) axis into the unit circle. However, this selection is a difficult problem in general.

Impulse invariance

A digital IIR filter can be obtained by sampling the impulse response of the analog prototype

\[ g_n \equiv h(nT) \]  

(10.6.1)

which may lead to aliasing in the frequency domain \( \Rightarrow \) chose small \( T \).

Assuming that the analog filter having \( N \) distinct poles is specified by:

\[ H_a(s) = \sum_{i=1}^{N} \frac{A_i}{s - \sigma_i} \]  

(10.6.2)

\[ h(t) = \sum_{i=1}^{N} A_i e^{\sigma_i t} u(t) \]  

(10.6.3)

Periodical sampling will lead to:

\[ g_n \equiv h_i(nT) = \sum_{i=1}^{N} A_i e^{\sigma_i nT} u_n \]  

(10.6.4)
Impulse invariance

The resulting digital filter will be:

\[ H(z) = \sum_{n=0}^{\infty} h_n z^{-n} = \sum_{n=0}^{\infty} \left( \sum_{i=1}^{N} A_i e^{\sigma_i n T} \right) z^{-n} = \sum_{i=1}^{N} A_i \sum_{n=0}^{\infty} \left( e^{\sigma_i T} z^{-1} \right)^n \]

(10.7.1)

The inner sum converges since \( \sigma_i < 0 \)

\[ \sum_{n=0}^{\infty} \left( e^{\sigma_i T} z^{-1} \right)^n = \frac{1}{1-e^{\sigma_i T} z^{-1}} \]

(10.7.2)

Therefore:

\[ H(z) = \sum_{i=1}^{N} A_i \frac{1}{1-e^{\sigma_i T} z^{-1}} \]

(10.7.3)

Digital poles at: \( p_i = e^{\sigma_i T} \Rightarrow |p_i| < 1 \Rightarrow \text{BIBO} \)

(10.7.4)

Not the best method due to aliasing. Also, impulse response can be infinite…

Bilinear transform

The techniques described so far, have severe limitations since they are appropriate only for LPFs and some BPFs. The bilinear transform does not have such limitations.

The bilinear transform from the \( s \)-plane to the \( z \)-plane is derived via the trapezoidal numerical integration of the differential equation describing the analog prototype. For the given step size \( T \), the BT is given by:

\[ s = \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}} \]

(10.8.1)

This transform is a one-to-one mapping; that is, it maps a single point in the \( s \)-plane to a unique point in the \( z \)-plane, and vice versa. The digital transfer function:

\[ G(z) = H_a(s) \bigg|_{s = \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}} \]

(10.8.2)
Bilinear transform

We can derive from (10.8.1) that

\[
\frac{z - 1 + \frac{s T}{2}}{1 - \frac{s T}{2}} = \frac{1 + \frac{T}{2} (\sigma + j \Omega)}{1 - \frac{T}{2} (\sigma + j \Omega)} = 1 + \frac{T}{2} \left( \frac{\sigma T}{2} + j \frac{\Omega T}{2} \right) \frac{1 - e + j x}{1 - e - j x}
\]

(10.9.1)

In this situation: \(|z| < 1 \Rightarrow \text{BIBO}\)

Similarly, when \(\sigma > 0, |z| > 1\); therefore, the mapping is correct.

Bilinear transform

When \(z = e^{j \omega}\):

\[
\xi = \frac{2}{T} \left( 1 - z^{-1} \right) = 2 \left( 1 - e^{-j \omega} \right) = 2 \frac{e^{-j \omega}}{T} \left[ e^{j \omega/2} - e^{-j \omega/2} \right] = \frac{2}{T} \left[ e^{j \omega/2} + e^{-j \omega/2} \right] = 2 \frac{j \sin \omega/2}{T} = j \frac{2}{T} \tan \left( \frac{\omega/2}{2} \right) = j \Omega
\]

(10.10.2)

Therefore:

\[
\frac{\Omega T}{2} = \tan \frac{\omega}{2}
\]

(10.10.3)

We can notice that \(\frac{\Omega_T}{\Omega} \neq \frac{\omega}{\omega_T}\) therefore, a "frequency warping" (10.10.3) is needed.
Bilinear transform

The BT destroys the phase response of an analog filter but preserves linear magnitude.

Therefore, to design a digital filter meeting the given specs for the magnitude response, we need to:

1) prewarp the critical bandage frequencies (\( \omega_p \) and \( \omega_s \)) to find their analog equivalents (\( \Omega_p \) and \( \Omega_s \)) using (10.10.3);
2) design the analog prototype using the prewarped critical frequencies;
3) apply the BT to obtain the desired digital transfer function.

Note: \( \delta_{p,a} = \delta_{p,d}, \delta_{s,a} = \delta_{s,d} \) (10.11.1)

BT maps the point \( s = \infty \) into the point \( z = -1 \).
BT is very good for LPF, HPF, BPF, BSF... – piecewise constant magnitude filters.

Bilinear transform – zero/pole conversion

Assuming that the analog zeros and poles are known: i.e. the analog transfer function is in the form:

\[
H(s) = k \frac{\prod_{n=1}^{\mu} (s - \sigma_n)}{\prod_{k=1}^{\nu} (s - \rho_k)}
\]  

(10.12.1)

Applying the BT:

\[
H(z) = k T^{\nu - \mu} \prod_{n=1}^{\mu} \left( 1 + \frac{2 - \sigma_n T}{2 - \sigma_n T} \right) \prod_{k=1}^{\nu} \left( 1 - \frac{2 + \rho_k T}{2 - \rho_k T} \right)
\]

(10.12.2)

New zeros

Constant: new gain factor

\( N-M \) zeros at \( z = -1 \)
(from zeros at \( \infty \))

New poles
Bilinear transform – zero/pole conversion

Therefore:

Digital gain:

\[ k_d = k_w \frac{\prod_{m=1}^{M} (2 - \sigma_m T)}{T^{N-M} \prod_{k=1}^{K} (2 - \rho_k T)} \quad (10.13.1) \]

Digital poles:

\[ p_k = \frac{2 + \rho_k T}{2 - \rho_k T} \quad (10.13.2) \]

Digital zeros:

\[ z_m = \frac{2 + \sigma_m T}{2 - \sigma_m T} \quad (10.13.3) \]

Need to add \( N-M \) zeros at \( z = -1 \):

“Theoretical” Example

Example 10.1: Design a single-pole digital LPF with a 3-dB bandwidth of 0.2\( \pi \), by use of the BT applied to the analog filter with the 3-dB bandwidth \( \Omega_c \):

\[ H(s) = \frac{\Omega_c}{s + \Omega_c} \quad (10.14.1) \]

Using the frequency warping:

\[ \Omega_c = \frac{2}{T} \tan \left( \frac{\varphi_s}{2} \right) = \frac{2}{T} \tan \left( 0.1 \pi \right) = \frac{0.65}{T} \quad (10.14.2) \]

The analog filter:

\[ H(s) = \frac{0.65/T}{s + 0.65/T} = \frac{0.65}{sT + 0.65} \quad (10.14.3) \]

Applying BT:

\[ H(z) = \frac{0.65}{2 - \frac{1 - z^{-1}}{1 + z^{-1}} + 0.65} = \frac{0.325}{1 - z^{-1} + 0.325z^{-1} + 0.325z^{-2}} = \frac{0.245 \cdot (1 + z^{-1})}{1 - 0.509z^{-1}} \quad (10.14.4) \]

Which leads to:

\[ H(e^{j\omega}) = \frac{0.245 \cdot (1 + e^{-j\omega})}{1 - 0.509e^{-j\omega}} \quad (10.14.5) \]
“Practical” Example

Example 10.2: Design a digital LPF starting from the elliptic prototype. The specs for the digital filter are:

\[
\begin{align*}
    f_p &= \frac{\omega_p}{2\pi} = 1000 \text{ Hz}; \\
    f_m &= \frac{\omega_m}{2\pi} = 1500 \text{ Hz}; \\
    f_s &= 10 \text{ kHz} \\
    1 - \delta_p &= -0.3 \text{ dB}; \\
    \delta_m &= -50 \text{ dB}
\end{align*}
\]

Note: \( T = \frac{1}{f_s} = 10^{-4} \text{ s} \)

The critical frequencies normalized for \( T = 1 \):

\[
\begin{align*}
    \omega_p &= 2\pi \frac{f_p}{f_s} = 0.2\pi; \\
    \omega_m &= 2\pi \frac{f_m}{f_s} = 0.3\pi; \\
    \frac{\omega}{\omega_p} &= 1.5
\end{align*}
\]

1. Frequency pre-warping (for \( T = 1 \)):

\[
\Omega = \frac{2}{T} \tan \frac{\omega}{2} \Rightarrow \Omega_p = 0.6498 \text{ rad/s}; \quad \Omega_m = 1.01905 \text{ rad/s}; \quad \Rightarrow \frac{\Omega_p}{\Omega_m} = 1.568
\]

2. Table lookup (software):

For the specs (actually, slightly higher than given ones):

\[
1 - \delta_p = -0.28 \text{ dB}; \quad \delta_m = -50.1 \text{ dB}; \quad N = 5; \quad \Omega_p/\Omega_m = 1.566
\]

we find the poles and zeros for \( \Omega_p = 1 \text{ rad/s} \):

\[
\begin{align*}
    \rho_{1,2} &= -0.09699 \pm j1.0300 \\
    \rho_{3,4} &= -0.33390 \pm j0.7177 \\
    \rho_5 &= -0.49519 \\
    \sigma_{1,2} &= \pm j1.6170 \\
    \sigma_{3,4} &= \pm j2.4377
\end{align*}
\]

Slightly different from Matlab

Stable – all poles are in the left half of the plane.
**“Practical” Example**

3. De-normalize the analog design:
   
   Multiply zeros and poles by $\Omega_p$:
   
   $\tilde{\rho}_{1,2} = \rho_{1,2} \cdot \Omega_p = -0.0630 \pm j0.6694$;  
   $\tilde{\rho}_{3,4} = \rho_{3,4} \cdot \Omega_p = -0.02170 \pm j0.4664$;  
   $\tilde{\sigma}_{1,2} = \sigma_{1,2} \cdot \Omega_p = \pm j1.0508$;  
   $\tilde{\sigma}_{3,4} = \sigma_{3,4} \cdot \Omega_p = \pm j1.5842$

4. BT transform of poles and zeros:

   $p_k = \frac{2 + \tilde{\rho}_k T}{2 - \tilde{\rho}_k T}$  
   
   $z_m = \frac{2 + \tilde{\sigma}_m T}{2 - \tilde{\sigma}_m T}$

Since $N-M = 1$, we must add one zero:

$z_5 = -1$

5. Verification

Need to check whether specs are satisfied.

Yes: we are done!  
No: start over (perhaps, a higher order of the prototype will help)

In general (especially, for more complicated filters), multiple iterations may be needed to satisfy specifications (meet or exceed them).
Frequency (spectral) transformation

It is often needed to modify the characteristics of existing digital filter to meet new specs without starting the design from scratch.

Frequency transformation can convert digital LPF to either another LPF, or HPF, or BPF, or BSF. The transformation involves replacing the variable \( z \) by a rational function \( F(\hat{z}) \), while satisfying the following conditions to preserve BIBO:

\[
|z| > 1 \Rightarrow |\hat{z}| < 1
\]  

(10.19.1)

Note: \( F(z) \) is an allpass filter:

\[
\frac{1}{1 - \alpha e^{-j\omega}} = \frac{1}{1 - a \hat{z}} \Rightarrow \hat{z} = \frac{z - a}{1 - a z^{-1}}
\]

(10.19.2)

Here \(|\alpha| < 1\) to ensure that a stable filter is transferred to another stable filter.

Frequency (spectral) transformation

1. LPF to LPF:

\[
z^{-1} = F^{-1}(\hat{z}) = \frac{1 - \alpha \hat{z}}{\hat{z} - a} = \frac{z^{-1} - a}{1 - a z^{-1}}
\]

(10.20.1)

\[
\Rightarrow e^{-j\omega} = \frac{e^{-j\hat{\omega}} - \alpha}{1 - \alpha e^{-j\hat{\omega}}}
\]

(10.20.2)

Where \( \omega \) is the “old frequency” and \( \hat{\omega} \) is the “new frequency”.

\[
e^{-j\omega} - \alpha e^{-j\hat{\omega}} = e^{-j\hat{\omega}} - \alpha \Rightarrow \alpha = \frac{e^{-j\hat{\omega}} - e^{-j\omega}}{1 - e^{-j\omega}} = \frac{e^{-j\hat{\omega}}}{e^{-j\hat{\omega}} e^{-j\omega}} \left( e^{j\hat{\omega}} - e^{j\omega} \right) = \frac{\sin \left( \frac{\omega + \hat{\omega}}{2} \right)}{\sin \left( \frac{\omega - \hat{\omega}}{2} \right)}
\]

(10.20.3)

(10.20.4)

We can relate ANY frequency component of the new LPF to the frequency component of the prototype LPF.
**Frequency (spectral) transformation**

Usually:

\[
\alpha = \frac{\sin \left( \frac{\omega_c - \tilde{\omega}}{2} \right)}{\sin \left( \frac{\omega_c + \tilde{\omega}}{2} \right)}
\]  

(10.21.1)

From (10.20.2):

\[
e^{-j\tilde{\omega}} = \frac{\alpha + e^{-j\omega}}{1 + \alpha e^{-j\omega}} = \frac{(\alpha + e^{-j\omega})(1 + \alpha e^{-j\omega})}{\left|1 + \alpha e^{-j\omega}\right|^2} = \frac{2\alpha + e^{-j\omega} + \alpha^2 e^{j\omega}}{\left|1 + \alpha e^{-j\omega}\right|^2}
\]

(10.21.2)

\[
\tilde{\omega} = -\tan^{-1}\left( \frac{\left[\alpha^2 - 1\right]\sin \omega}{2\alpha + \left[\alpha^2 + 1\right]\cos \omega} \right)
\]

(10.21.3)

Clearly:

\[
\frac{\omega_c}{\omega_p} \neq \frac{\tilde{\omega}_1}{\tilde{\omega}_p}
\]

(10.21.4)

**Frequency (spectral) transformation**

Example of mapping:

Note: the gain in unchanged
Frequency (spectral) transformation

2. LPF to HPF:

\[ z^{-1} = F^{-1}(\tilde{z}) = \frac{1 + \alpha z^{-1}}{1 + \alpha} \] (10.23.1)

\[ \Rightarrow e^{-j\omega} = \frac{1}{1 + \alpha e^{-j\omega}} \] (10.23.2)

\[ e^{-j\omega} + \alpha e^{-j\omega} = -e^{-j\omega} - \alpha \] (10.23.3)

\[ \alpha = \frac{-e^{-j\omega} - e^{-j\omega}}{1 + e^{-j\omega} e^{-j\omega}} = \frac{\cos \left( \frac{\omega - \tilde{\omega}}{2} \right) \omega + \tilde{\omega}}{\cos \left( \frac{\omega + \tilde{\omega}}{2} \right)} \] (10.23.4)

\[ \tilde{\omega} = -\tan^{-1} \left( \frac{\left[ 1 - \alpha^2 \right] \sin \omega}{-2\alpha - \left[ \alpha^2 + 1 \right] \cos \omega} \right) \] (10.23.5)

Example of mapping:

![Frequency mapping graph](image-url)
Frequency (spectral) transformation

3. LPF to BPF (filter order will be doubled):

\[
\begin{align*}
\beta^{-1} z^{-2} - 2 \frac{\alpha \beta}{\beta + 1} z^{-1} + \frac{\beta - 1}{\beta + 1} \\
\beta^{-1} z^{-2} - 2 \frac{\alpha \beta}{\beta + 1} z^{-1} + 1
\end{align*}
\]

\[
\alpha = \frac{\cos \left( \frac{\tilde{\omega}_2 + \tilde{\omega}_1}{2} \right)}{\cos \left( \frac{\tilde{\omega}_2 - \tilde{\omega}_1}{2} \right)}
\]

\[
\beta = \cotan \left( \frac{\tilde{\omega}_2 - \tilde{\omega}_1}{2} \right) \tan \left( \frac{\omega_c}{2} \right)
\]

Here \( \tilde{\omega}_2 \) and \( \tilde{\omega}_1 \) are new upper and lower cutoff frequencies.

A special case: bandwidth preserving

\[
\tilde{\omega}_2 - \tilde{\omega}_1 = \omega_c
\]

Therefore:

\[
\beta = 1
\]

and:

\[
\begin{align*}
\beta^{-1} z^{-2} - \alpha z^{-1} \\
-\alpha z^{-1} + 1
\end{align*}
\]

\[
\beta^{-1} z^{-2} - \alpha z^{-1} = \beta^{-1} z^{-2} - \alpha z^{-1}
\]

\[
\begin{align*}
\beta^{-1} z^{-2} - \alpha z^{-1} \\
-\alpha z^{-1} + 1
\end{align*}
\]

\[
\beta^{-1} z^{-2} - \alpha z^{-1} = \beta^{-1} z^{-2} - \alpha z^{-1}
\]
Frequency (spectral) transformation

4. LPF to BSF (filter order will be doubled):

\[
z^{-1} = \frac{z^{-2} - 2 + \frac{1 - \beta}{1 + \beta} + \frac{1 - \beta}{1 + \beta}z^{-1}}{1 - \beta z^{-2} - 2 + \frac{1 - \beta}{1 + \beta} z^{-1} + 1}
\]

(10.27.1)

\[
\alpha = \frac{\cos \left( \frac{\tilde{\omega}}{2} \right)}{\cos \left( \frac{\tilde{\omega}}{2} \right)}
\]

(10.27.2)

\[
\beta = \tan \left( \frac{\tilde{\omega}}{2} \right) \tan \left( \frac{\omega}{2} \right)
\]

(10.27.3)

Spectral transformation: Example

Example 10.3: Design a BW preserving BPF from the PL prototype:

\[
G(z) = \frac{1 + z^{-1}}{1 - 0.9z^{-1}} = \frac{z + 1}{z - 0.9}
\]

(10.28.1)

DC gain: 

\[
G(z) \big|_{z=e^{j\omega}} = \frac{2}{1 - 0.9} = 20 \quad (\approx 26 \, dB)
\]

(10.28.2)
Spectral transformation: Example

Since the filter is BW is preserving, $\beta = 1$

$$z^{-1} = -z^{-1} \left( \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}} \right) \Rightarrow e^{-j\omega} = -e^{-j\omega} \left( \frac{e^{-j\omega} - \alpha}{1 - \alpha e^{-j\omega}} \right)$$

For $\omega = 0$:

$$1 = -e^{-j\omega} \left( \frac{e^{-j\omega} - \alpha}{1 - \alpha e^{-j\omega}} \right)$$

$$\Rightarrow (1 - \alpha e^{-j\omega}) e^{j\omega} = \alpha - e^{-j\omega} \Rightarrow e^{-j\omega} - \alpha = \alpha - e^{-j\omega} \Rightarrow \alpha = \cos \omega$$

Assuming $\alpha = 0.4\pi$ - this is where we map DC. $\Rightarrow \alpha = 0.309016994$

In a general case, both $\alpha$ and $\beta$ can be found from (10.27.2) and (10.27.2).

$$z^{-1} = F^{-1}(\tilde{z}) = -\tilde{z}^{-1} \left( \frac{\tilde{z}^{-1} - \alpha}{1 - \alpha \tilde{z}^{-1}} \right) \quad F(\tilde{z}) = \frac{1 - \alpha \tilde{z}^{-1}}{\tilde{z}^{-1} - \alpha} = \frac{\tilde{z}(\tilde{z} - \alpha)}{1 - \alpha \tilde{z}}$$

Spectral transformation: Example

Method 1:

$$G_{LP}(z) = \frac{1 + z^{-1}}{1 - 0.9 z^{-1}} \rightarrow G_{BP}(z)$$

$$G_{BP}(z) = \frac{1 + z^{-1} \left( \frac{\alpha - z^{-1}}{1 - \alpha z} \right)}{1 - 0.9 z^{-1} \left( \frac{\alpha - z^{-1}}{1 - \alpha z} \right)} = \frac{(1 - \alpha z^{-1} + \alpha z^{-1} - z^{-1})(1 - az^{-1})}{(1 - az^{-1})(1 - az^{-1} - 0.9az^{-1} + 0.9z^{-2})}$$

$$= \frac{1 - z^{-2}}{1 - 1.9az^{-1} + 0.9z^{-2}}$$

Method 2: SFG replacement...
Spectral transformation: Example

Method 3 (preferred):

Zeros: \[ z_1 = F(\tilde{z}_1) \Rightarrow -1 = -\frac{\tilde{z}(\tilde{z} - \alpha)}{1 - \alpha \tilde{z}} \Rightarrow -1 + \alpha \tilde{z} = -\tilde{z}^2 + \alpha \tilde{z} \]
\[ \Rightarrow \tilde{z}^2 = 1 \Rightarrow \tilde{z}_{1,2} = \pm 1 \]

Poles: \[ p_k = F(\tilde{p}_k) \Rightarrow 0.9 = -\frac{\tilde{p}(\tilde{p} - \alpha)}{1 - \alpha \tilde{p}} \Rightarrow 0.9 - 0.9\alpha \tilde{p} = -\tilde{p}^2 + \alpha \tilde{p} \]
\[ \Rightarrow \tilde{p}^2 - 1.9\alpha \tilde{p} + 0.9 = 0 \]
\[ \Rightarrow \tilde{p}_{1,2} = \frac{1.9\alpha \pm \sqrt{1.9^2\alpha^2 - 4 \cdot 0.9}}{2} \Rightarrow \tilde{p}_{1,2} = 0.2936 \pm j0.9021 \]

Therefore: \[ G_{BP}(z) = k \frac{1 + z^{-2}}{1 - 1.9\alpha z^{-1} + 0.9z^{-2}} = \{k = 1\} = \frac{1 + z^{-2}}{1 - 0.5871z^{-1} + 0.9z^{-2}} \]

The new gain can be determined experimentally...